On exit times of Lévy-driven Ornstein–Uhlenbeck processes

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Abstract

We prove two martingale identities which involve exit times of Lévy-driven Ornstein–Uhlenbeck processes. Using these identities we find an explicit formula for the Laplace transform of the exit time under the assumption that positive jumps of the Lévy process are exponentially distributed.

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1 Introduction

Let X_t, $t \geq 0$, be an Ornstein–Uhlenbeck (O-U) process driven by a Lévy process L_t, i.e. X_t is a solution of the equation

$$X_t = x - \beta \int_0^t X_s ds + L_t, \quad t \geq 0.$$

We assume that the parameter β is positive and the initial value $X_0 = x$ is non-random.

In the special case when L_t is a compound Poisson process, the process X_t is also known in applications as a "shot-noise" process or a "storage process" with a linear release function.

One of the most important for the models of that sort problems is to determine or to approximate the distribution of the first passage time

$$\tau_b = \inf\{t > 0 : X_t \geq b\}$$

of a given level $b > x$. The problem was discussed for the Gaussian O-U processes by Darling & Siegert (1953). Explicit representations for the Laplace

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transform $E(e^{-\mu\tau_b})$ were found in the papers of Hadjiev (1983) and Novikov (1990, 2003) in the case when L_t has no positive jumps (the so-called spectrally negative case). Moreover, the papers of Novikov and Èrgashev (1993) and Novikov (2003) provide some bounds and asymptotic approximations for the distribution of τ_b. In particular, it was proved in Novikov (2003), Theorem 2, that the distribution of τ_b is exponentially bounded under the condition that L_t has a diffusion part or positive jumps. The papers Perry et al. (2001) and Borovkov and Novikov (2001) contain some general results on integral equations for the distributions of τ_b and X_{τ_b}.

It seems that the first results for Lévy-driven O-U processes with exponentially distributed jumps were obtained by Tsurui and Osaki (1976) for the case when the parameter $1/\beta$ is an integer. For the case of arbitrary $\beta > 0$ and exponentially distributed positive jumps, explicit formulas for the Laplace transform $E(e^{-\mu\tau_b})$ and the expectation of τ_b can be found in the paper of Novikov et al. (2005) who solved the corresponding integro-differential equation. Recently, Jacobsen and Jensen (2006) found the joint Laplace transform $E(e^{-\mu\tau_b + wX_{\tau_b}})$ in the form of a linear combination of contour integrals under the assumption that a distribution of the positive jumps is a mixture of exponential ones.

In what follows, we always assume that the following condition holds:

$$E\log(1 + |L_1|) < \infty \tag{1}$$

(this is a sufficient and necessary condition for convergence of X_t in distribution to a proper limit, see e.g. Wolfe (1982)).

In Section 2 we prove two martingale identities (see Theorems 1 and 2 below) which involve both the first passage time τ_b and X_{τ_b}. These identities enable one to obtain explicit bounds for the distribution of τ_b (e.g. an explicit lower bound for $E\tau_b$) just by neglecting the overshoot

$$\chi_b = X_{\tau_b} - b.$$

In Section 3 we use Theorems 1 and 2 for deriving explicit representations of the Laplace transform and the mean of τ_b under the assumption that the positive jumps of L_t are exponentially distributed but without any restrictions on the distribution of the negative jumps of L_t. We also prove the Exponential Limit Theorem for τ_b as $b \to \infty$.

2 Martingale identities

In what follows we always assume that L_t has a non-zero component with positive jumps(or, equivalently, $\Pi(0, \infty) > 0$ where $\Pi(dx)$ is the Lévy-Khinchin measure associated with L_t). This assumption implies that L_t has the following representation:

$$L_t = Q_t + R_t \tag{2}$$

with the compound Poisson process

$$R_t = \sum_{k=1}^{N_t} \xi_k, \tag{3}$$

where ξ_k are the jumps of L_t which are greater than some positive number δ,
$P\{\xi_k > \delta\} > 0$, N_t is a Poisson process with rate $E(N_1) = \lambda > 0$. We also assume that the componet Q_t can only contain a diffusion part and jumps less than or equal to δ and therefore Q_t and R_t are independent.

Set

$$K = \sup\{u \geq 0 : Ee^{uL_1} < \infty\}.$$

We shall further assume that $K > 0$ and set

$$\varphi(u) = \frac{1}{\beta} \int_0^u \frac{\log(Ee^{vL_1})}{v} dv, \quad 0 \leq u < K. \tag{4}$$

Since Q_t and R_t are independent, we have

$$\varphi(u) = \Delta(u) + W(u), \tag{5}$$

where we put

$$\Delta(u) = \frac{1}{\beta} \int_0^u \frac{\log(Ee^{vQ_1})}{v} dv, \tag{6}$$

$$W(u) = \frac{1}{\beta} \int_0^u \frac{\log(Ee^{vR_1})}{v} dv = \frac{\lambda}{\beta} \int_0^u \frac{(Ee^{v\xi_1} - 1)}{v} dv.$$

Under the assumption (1) the integrals in (4) and (6) converge (see some details of the proof for this fact in Wolfe (1982) or Novikov (2003)) and so $\varphi(u)$, $\Delta(u)$ and $W(u)$ are finite continuous functions. Besides, for all $u \in [0, \infty)$ the following lower bound holds[1]

$$\Delta(u) \geq -c - Cu \tag{7}$$

(see Novikov (2003)).

Using the inequality $e^x > 1 + x + x^2/2$, $x > 0$, we obtain also

$$W(u) \geq \frac{\lambda}{\beta}(u\delta + u^2\delta^2/4)P\{\xi_k > \delta\} > 0. \tag{8}$$

Set

$$\varphi(K) = \lim_{u \uparrow K} \varphi(u).$$

If $K = \infty$, then $\varphi(K) = \infty$ due to (8). If $0 < K < \infty$, then the value $\varphi(K)$ could be finite or infinite as illustrated by the following example where the

[1] c and C are some positive constants

Compound Poisson process $L_t = \sum_{k=1}^{N_t} \xi_k,$ has the jumps ξ_k with the Gamma distribution, i.e.

$$P(\xi_k \in dx) = \frac{x^{\rho-1}e^{-x}}{\Gamma(\rho)}dx,\ x > 0, \rho > 0.$$

Then $K = 1$ and by direct calculation

$$\varphi(u) = \frac{\lambda}{\beta}\int_0^u \frac{1-(1-v)^\rho}{(1-v)^\rho v}dv, \quad u < 1,$$

so that

$$\varphi(1) < \infty \quad for \ \ \rho < 1$$

and

$$\varphi(1) = \infty \quad for \ \ \rho \geq 1.$$

Note that when $\rho = 1$ (this is the case of exponentially distributed jumps with mean one) we have the explicit formula

$$\varphi(u) = -\frac{\lambda}{\beta}\log(1-u), \quad u < 1. \tag{9}$$

Set

$$G(z,\mu) = \int\limits_0^K e^{uz-\varphi(u)}u^{\mu-1}du, \quad \mu > 0. \tag{10}$$

This function is, obviously, finite when $K < \infty$. For the case $K = \infty$ the finiteness of $G(z,\mu)$ is implied by (7) and (8).

Theorem 1. *Let condition* (1) *hold,* $0 < K \leq \infty$ *and* $\varphi(K) = \infty$. *Then*

$$E(e^{-\mu\beta\tau_b}G(X_{\tau_b},\mu)) = G(x,\mu), \quad \mu > 0. \tag{11}$$

Proof. First consider the case

$$K = \infty,$$

in which it was shown by Novikov (2003) that the process $e^{-\mu\beta t}G(X_t,\mu)$ is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma\{L_s, s \leq t\}$.

Applying the optional stopping theorem, we have for any $t > 0$

$$E[I\{\tau_b \leq t\}e^{-\mu\beta\tau_b}G(X_{\tau_b},\mu)] + E[I\{\tau_b > t\}e^{-\mu\beta t}G(X_t,\mu)] = G(x,\mu). \tag{12}$$

Since $X_t \leq b$ on the event $\{\tau_b > t\}$ and $G(x,\mu)$ is a nondecreasing function of x, we obtain

$$E[I\{\tau_b > t\}e^{-\mu\beta t}G(X_t,\mu)] \leq e^{-\mu\beta t}P\{\tau_b > t\}G(b,\mu) \to 0$$

as $t \to \infty$. The first term on the LHS of (12) clearly converges monotonically to $E(e^{-\mu\beta\tau_b}G(X_{\tau_b},\mu))$ as $t \to \infty$ because τ_b is finite with probability one (in fact, it is even exponentially bounded). So (11) holds when $K = \infty$.

To prove (11) in the case $0 < K < \infty$, we shall truncate positive jumps of L_t by a positive constant A and then justify a passage to the limit as $A \to \infty$.

Set

$$L_t^A = Q_t + R_t^A$$

with

$$R_t^A = \sum_{k=1}^{N_t} \min(\xi_k, A).$$

Let X_t^A be an O-U process driven by L_t^A,

$$\tau_b^A = \inf\{t \geq 0 : \ X_t^A \geq b\}$$

and

$$\varphi_A(u) = \frac{1}{\beta} \int_0^u \frac{\log(Ee^{vL_1^A})}{v} dv = \Delta(u) + W(u, A),$$

where we put

$$W(u, A) = \frac{\lambda}{\beta} \int_0^u \frac{(Ee^{v \min(\xi_1, A)} - 1)}{v} dv. \tag{13}$$

It is obvious from the Lévy-Khintchin formula that the right distribution tail of L_1^A decays faster than any exponential function, so that the respective value $K = K(A) = \infty$. Hence, identity (11) does hold for the process X_t^A :

$$E\left[e^{-\mu\beta\tau_b^A} \int_0^\infty \exp\{uX_{\tau_b^A}^A - \varphi_A(u)\} u^{\mu-1} du\right] = \int_0^\infty e^{ux - \varphi_A(u)} u^{\mu-1} du. \tag{14}$$

Further we note that as $A \to \infty$

$$\int_0^\infty e^{ux - \varphi_A(u)} u^{\mu-1} du \to \int_0^K e^{ux - \varphi(u)} u^{\mu-1} du \tag{15}$$

which gives the RHS of (11). To see this, we note that, as $A \to \infty$, for any $u < K$

$$W(u, A) \to W(u), \quad \varphi_A(u) \to \varphi(u),$$

and for $u \geq K$

$$W(u, A) \to \infty.$$

and, oviously, the last two relations imply (15). Next, on the LHS of (14) we

write

$$\int_0^\infty = \int_0^K + \int_K^\infty$$

and consider convergence of the corresponding two terms separately. Note that

$$X_{\tau_b}^A \leq b + \delta + \min(\xi_{N_{\tau_b^A}}, A). \tag{16}$$

Obviously, τ_b^A could only decrease as A increases. Choose now a positive constant A_0 such that $P\{\xi_1 < A_0\} > 0$ and so $\tau_b^{A_0}$ is exponentially bounded. Then we have for all $A > A_0$

$$e^{uX^A_{\tau_b}} \le e^{u(b+\delta)} \sum_{k=1}^{N_{\tau_b^{A_0}}} e^{u\min(\xi_k, A)} \tag{17}$$

where by Wald's identity

$$E(\sum_{k=1}^{N_{\tau_b^{A_0}}} e^{u\min(\xi_k, A)}) = E(N_{\tau_b^{A_0}})E(e^{u\min(\xi_1, A)}) \tag{18}$$

and

$$E(N_{\tau_b^{A_0}}) = \lambda E(\tau_b^{A_0}) < \infty.$$

Collecting together the above bounds we obtain for the $\int\limits_K^\infty$ −part of the LHS of (14) the following bound:

$$E\left[e^{-\mu\beta\tau_b^A}\int\limits_K^\infty \exp\{uX^A_{\tau_b^A} - \varphi_A(u)\}u^{\mu-1}du\right]$$
$$\le C\int\limits_K^\infty E(e^{u\min(\xi_1,A)})e^{u(b+\delta)-\varphi_A(u)}u^{\mu-1}du,$$

where $C = \lambda E(\tau_b^{A_0})$. To show that the last integral converges to zero as $A \to \infty$, we note that (13) implies

$$E(e^{u\min(\xi_1,A)}) = \frac{\beta u}{\lambda}\frac{\partial W(u,A)}{\partial u} + 1.$$

This means that

$$\int\limits_K^\infty E(e^{u\min(\xi_1,A)})e^{u(b+\delta)-\varphi_A(u)}u^{\mu-1}du$$
$$= \frac{\beta}{\lambda}\int\limits_K^\infty \frac{\partial W(u,A)}{\partial u}e^{u(b+\delta)-W(u,A)-\Delta(u)}u^{\mu}du + \int\limits_K^\infty e^{u(b+\delta)-W(u,A)-\Delta(u)}u^{\mu-1}du. \tag{19}$$

The last integral tends to zero as $A \to \infty$ due to the fact that $W(u,A) \to \infty$ for $u \ge K$.

Integrating by parts the first integral on the RHS of (19), we obtain:

$$\int\limits_K^\infty \frac{\partial W(u,A)}{\partial u}e^{u(b+\delta)-W(u,A)-\Delta(u)}u^{\mu}du = -\int\limits_K^\infty e^{u(b+\delta)-\Delta(u)}u^{\mu}d(e^{-W(u,A)})$$

$$= e^{K(b+\delta)-\Delta(K)} K^\mu e^{-W(K,A)} + \int_K^\infty e^{-W(u,A)} d(e^{u(b+\delta)-\Delta(u)} u^\mu).$$

Now it should be clear that the last two terms converge to zero as $A \to \infty$ due to the fact that $W(u, A) \to \infty$ for $u \geq K$. So, we have proved the convergence of the $\int_K^\infty$ −part of the LHS of (14) to zero.

To study the part with $\int_0^K$ on the LHS of (14), note that the random variable $X^A_{\tau^A_b}$ coincides with X_{τ_b} on the set $\{\max_{k \leq N_{\tau^A_b}} \xi_k < A\}$ (because no jumps are truncated up to the time $\tau_b \leq \tau^A_b$). Obviously, as $A \to \infty$

$$P\{\max_{k \leq N_{\tau^A_b}} \xi_k < A\} \to 1, \tag{20}$$

and hence

$$E\left[I\{\max_{k \leq N_{\tau^A_b}} \xi_k < A\} e^{-\mu\beta\tau^A_b} \int_0^K e^{uX_{\tau_b} - \varphi_A(u)} u^{\mu-1} du \right] \\ \to E\left[e^{-\mu\beta\tau_b} \int_0^K e^{uX_{\tau_b} - \varphi(u)} u^{\mu-1} du \right].$$

To complete the proof, we need to check only that

$$\lim_{A\to\infty} E\left[I\{\max_{k \leq N_{\tau^A_b}} (\xi_k) \geq A\} \int_0^K e^{uX^A_{\tau^A_b} - \varphi_A(u)} u^{\mu-1} du \right] = 0. \tag{21}$$

To see this, note that in view of (17), we have for all $A \geq A_0$

$$\int_0^K e^{uX^A_{\tau^A_b} - \varphi_A(u)} u^{\mu-1} du \leq \sum_{k=1}^{N_{\tau^{A_0}_b}} \int_0^K e^{u \min(\xi_k, A)} e^{u(b+\delta)-\Delta(u)} e^{-W(u,A)} u^{\mu-1} du \leq C\eta_A,$$

where we put

$$\eta_A = \sum_{k=1}^{N_{\tau^{A_0}_b}} \int_0^K e^{u \min(\xi_k, A)} e^{-W(u,A)} u^{\mu-1} du, \; C = e^{K(b+\delta) - \min_{u \leq K} \Delta(u)}.$$

Due to this bound and (20), for the validity of (21) it is sufficient to show that $\{\eta_A, A \geq A_0\}$ is a family of uniformly integrable random variables or, equivalently, that

$$\lim_{A\to\infty} E(\eta_A) = E(\lim_{A\to\infty} \eta_A) < \infty.$$

In view of (18) the latter property is equivalent to

$$\lim_{A\to\infty} E(\int_0^K e^{u\min(\xi_k,A)}e^{-W(u,A)}u^{\mu-1}du) = E(\int_0^K e^{u\xi_k}e^{-W(u)}u^{\mu-1}du) < \infty.$$

which one can easily verify (e.g. using monotonicity of the functions $\min(\xi_k, A)$ and $W(u, A)$).

This completes the proof.

Theorem 2. *Let condition (1) hold, $0 < K < \infty$ and $\varphi(K) = \infty$. Then*

$$\beta E(\tau_b) = E\int_0^K (e^{uX_{\tau_b}} - e^{ux})e^{-\varphi(u)}u^{-1}du.$$

Proof. We will derive this identity from Theorem 1 by passing to the limit as $\mu \to 0$. To justify this procedure we observe that (11) can be written in the following form:

$$E(e^{-\mu\beta\tau_b}\int_0^K (e^{uX_{\tau_b}} - e^{ux})e^{-\varphi(u)}u^{\mu-1}du) = (1 - Ee^{-\mu\beta\tau_b})\int_0^K e^{ux-\varphi(u)}u^{\mu-1}du, \ \ \mu > 0.$$

Here the LHS converges to $E\int_0^K (e^{uX_{\tau_b}} - e^{ux})e^{-\varphi(u)}u^{-1}du$ as $\mu \to 0$. One can easily see (e.g. using integration by parts) that

$$\int_0^K e^{ux-\varphi(u)}u^{\mu-1}du \sim \frac{1}{\mu}. \tag{22}$$

This implies that

$$\lim_{\mu\to 0}(1 - Ee^{-\mu\beta\tau_b})\int_0^K e^{ux-\varphi(u)}u^{\mu-1}du = \beta E(\tau_b)$$

which completes the proof of Theorem 2.

Remark. The assertion of Theorem 2 was proved for the case $K = \infty$ in Novikov (2003) under an additional assumption that

$$E(L_1^-)^\delta < \infty \textit{ for some } \delta > 0.$$

3 Exponentially distributed positive jumps

In this section we use the same notation as in Section 2 and assume that the process Q_t in the decomposition (2) does not contain positive jumps while

the process R_t is a compound Poisson process with exponentially distributed positive jumps, $E(\xi_k) = K,\ 0 < K < \infty;$ N_t is a Poisson process with rate $E(N_1) = \lambda > 0$. Note that under these assumptions

$$e^{-\varphi(u)} = (1 - u/K)^{\lambda/\beta} e^{-\Delta(u)}.$$

Theorem 3. *For any $\mu > 0$,*

$$E(e^{-\mu\beta\tau_b}) = \frac{\int_0^K (1-u/K)^{\lambda/\beta} u^{\mu-1} e^{ux-\Delta(u)} du}{\int_0^K (1-u/K)^{\lambda/\beta-1} u^{\mu-1} e^{ub-\Delta(u)} du}, \tag{23}$$

$$E(\tau_b) = \frac{1}{\beta} \int_0^K (e^{ub} - e^{ux}(1-u/K))(1-u/K)^{\lambda/\beta-1} e^{-\Delta(u)} u^{-1} du. \tag{24}$$

Besides, as $b \to \infty$,

$$E(\tau_b) = Ce^{Kb}(Kb)^{-\lambda/\beta}(1 + o(1)),\ C = \frac{\Gamma(\lambda/\beta)}{\beta K} e^{-\Delta(K)} \tag{25}$$

and the Exponential Limit Theorem holds:

$$P\{\frac{\tau_b}{E(\tau_b)} > x\} \to e^{-x},\ x > 0. \tag{26}$$

Proof. Formulas (23) and (24) are direct consequences of Theorem 1, Theorem 2 and the following two well-known facts (which hold due to the memory-less property of the exponential distribution, see a similar statement in Borovkov (1976) for the case $\beta = 0$):

1) the overshoot $\chi_b = X_{\tau_b} - b$ has the density

$$p_{\chi_b}(x) = \frac{1}{K} e^{-x/K}, \quad x > 0;$$

2)

$$\chi_b \quad \textit{and} \quad \tau_b \ \textit{are independent.}$$

Relation (24) implies, using the change of variables $(1 - u/K)b = w$,

$$\frac{d}{db} E(\tau_b) = \frac{1}{\beta} \int_0^K e^{ub} (1-u/K)^{\lambda/\beta-1} e^{-\Delta(u)} du$$

$$= \frac{1}{\beta} e^{Kb} b^{-\lambda/\beta} \int_0^b e^{-Kw} w^{\lambda/\beta-1} e^{-\Delta(K(1-w/b))} dw.$$

Since the function $\Delta(K(1-w/b))$ is continuous and bounded in $w \in [0, K]$, the last integral converges as $b \to \infty$ to

$$e^{-\Delta(K)} \int_0^\infty e^{-Kw} w^{\lambda/\beta-1} dw = e^{-\Delta(K)} \Gamma(\lambda/\beta) K^{-\lambda/\beta}.$$

Hence

$$\frac{d}{db} E(\tau_b) \sim \frac{\Gamma(\lambda/\beta)}{\beta} e^{-\Delta(K)} e^{Kb} (bK)^{-\lambda/\beta}, \ b \to \infty.$$

By well-know facts of theory of asymptotic expansions (see e.g. Olver (1997)) it implies

$$E(\tau_b) \sim \frac{\Gamma(\lambda/\beta)}{\beta K} e^{-\Delta(K)} e^{Kb} (bK)^{-\lambda/\beta}$$

and therefore we have proved (25).

To derive (26), we write the denominator in (23) as follows:

$$\int_0^K (e^{ub} - e^{ux}(1-\frac{u}{K}))(1-\frac{u}{K})^{\lambda/\beta-1} u^{\mu-1} e^{-\Delta(u)} du + \int_0^K (1-\frac{u}{K})^{\lambda/\beta} u^{\mu-1} e^{ux-\Delta(u)} du. \tag{27}$$

Set

$$\mu = z/(\beta E\tau_b)$$

with a fixed $z > 0$ and $E(\tau_b)$ defined in (24). Clearly,$\mu \to 0$ as $b \to \infty$. Due to (22), the second term in (27) (which is also the nominator in (23)) can now be written as

$$\int_0^K (1-u/K)^{\lambda/\beta} u^{\mu-1} e^{ux-\Delta(u)} du = \frac{\beta E(\tau_b)}{z}(1+o(1)). \tag{28}$$

Using (24), the first term in (27) can be written as

$$\int_0^K (e^{ub} - e^{ux}(1-u/K))(1-u/K)^{\lambda/\beta-1} u^{\mu-1} e^{-\Delta(u)} du = \beta E(\tau_b) + \delta(b), \tag{29}$$

where

$$\delta(b) = \int_0^K (e^{ub} - e^{ux}(1-u/K))(1-u/K)^{\lambda/\beta-1} u^{-1}(u^\mu - 1) e^{-\Delta(u)} du.$$

Note that for $u > 0$

$$|u^\mu - 1| = |e^{\mu \log u} - 1| \le \mu \max(u^\mu, 1) |\log u|.$$

This implies

$$|\delta(b)| \leq \mu \max(K^{\mu}, 1) \int_{0}^{K} (e^{ub} - e^{ux}(1-u/K))(1-u/K)^{\lambda/\beta - 1} e^{-\Delta(u)} u^{-1} |\log u| du.$$

Applying the same change of variables as above, one can easily show that the last integral is $O(e^{Kb} b^{-\lambda/\beta})$ as $b \to \infty$.

Taking into account (25), due to the setting for μ we get

$$\delta(b) = O(1).$$

Now making the substitution in (23) relations (27), (28), (29) with the last result, we obtain

$$E(e^{-z\tau_b/E(\tau_b)}) = \frac{\frac{\beta E(\tau_b)}{z}(1+o(1))}{\beta E(\tau_b) + O(1) + \frac{\beta E(\tau_b)}{z}(1+o(1))} \to \frac{1}{z+1}.$$

Since the function $\frac{1}{1+z}$ is the Laplace transform of the exponential distribution with mean 1, this completes the proof.

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